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Multivariate complex B-splines and Dirichlet averages

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Abstract

The notion of a complex B-spline is extended to a multivariate setting by means of ridge functions employing the known geometric relationship between ordinary B-splines and multivariate B-splines. To derive properties of complex B-splines in $\mathbb{R}^s$, $1 < s \in \mathbb{N}$, the Dirichlet average has to be generalized to include infinite-dimensional simplices Δ^∞. Based on this generalization several identities of multivariate complex B-splines are presented.

Keywords: Complex splines; Multivariate splines; Dirichlet average; Weyl fractional derivative and integral

1. Introduction

Recently, a family of B-splines with complex orders was defined in [7] and some of their properties were discussed. Complex B-splines are a natural extension of so-called *fractional B-splines*, first investigated in [25]. In the definition of a complex B-spline one replaces the non-negative integer-valued order by a complex number. More precisely, let $\operatorname{Re} z > 1$ be fixed and define $B_z : \mathbb{R} \to \mathbb{C}$ by

$$B_z(x) := \frac{1}{\Gamma(z)} \sum_{k \geq 0} (-1)^k \binom{z}{k} (x - k)_+^{z-1}, \tag{1.1}$$

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where

$$x_+^z = \begin{cases} x^z = \mathrm{e}^{z \ln x}, & \text{if } x > 0, \\ 0, & \text{if } x \le 0. \end{cases}$$

Here, $\Gamma : \mathbb{C} \setminus \mathbb{Z}_0^- \to \mathbb{C}$ denotes the Euler gamma function.

The series (1.1) converges for all $x \in \mathbb{R}$. It has been shown that, for fixed z with $\operatorname{Re} z > 1$, the functions B_z are elements of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and that their Fourier transform is given by

$$\mathcal{F}(B_z)(\omega) = \int_{\mathbb{R}} B_z(x)\mathrm{e}^{-\mathrm{i}\omega x}\mathrm{d}x = \left(\frac{1-\mathrm{e}^{-\mathrm{i}\omega}}{\mathrm{i}\omega}\right)^z.$$

It thus follows that

$$\int_{\mathbb{R}} B_z(x)\,\mathrm{d}x = \mathcal{F}(B_z)(0) = 1. \tag{1.2}$$

For $z = n \in \mathbb{N}$, the complex B-spline B_z reduces to the classical Curry–Schoenberg B-spline $B_n, n \in \mathbb{N}$, [3].

For an interpretation of the complex B-spline in the Fourier domain, we set $\Omega(\omega) := \frac{1-\mathrm{e}^{-\mathrm{i}\omega}}{\mathrm{i}\omega}$. Then

$$\begin{aligned} \mathcal{F}(B_z)(\omega) &= (\Omega(\omega))^z = (\Omega(\omega))^{\operatorname{Re} z}\mathrm{e}^{\mathrm{i}\operatorname{Im} z \ln|\Omega(\omega)|}\mathrm{e}^{-\operatorname{Im} z \arg \Omega(\omega)} \\ &= \mathcal{F}(B_{\operatorname{Re} z})(\omega)\,\mathrm{e}^{\mathrm{i}\operatorname{Im} z \ln|\Omega(\omega)|}\,\mathrm{e}^{-\operatorname{Im} z \arg \Omega(\omega)}. \end{aligned}$$

In other words, a complex B-spline is a fractional B-spline of order $\operatorname{Re} z$ with an additional phase and a modulation/scaling factor in the Fourier domain. Note that because of $\arg \Omega(\omega) \ge 0$, the frequency components on the negative and positive axis are enhanced by the opposite sign. The fractional as well as the complex B-splines are scaling functions for multiresolution analyses of $L^2(\mathbb{R})$. (For details and constructions, we refer to [25] and [7].)

In [8], relations between complex B-splines and divided differences of complex order $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$ were derived. To this end, let $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ be interpreted as a sequence of uniform knots. We define the corresponding complex divided difference operator of order z by

$$[z; \mathbb{N}_0]g := \sum_{k\ge 0}(-1)^k \frac{g(k)}{\Gamma(z-k+1)\Gamma(k+1)}, \tag{1.3}$$

for all functions $g : \mathbb{R} \to \mathbb{C}$ with convergent series on the right hand side. Then

$$B_z(x) = z[z; \mathbb{N}_0](x - \bullet)_+^{z-1} \tag{1.4}$$

(cf. [8]). For $z = n \in \mathbb{N}_0$, Eqs. (1.3) and (up to a factor $(-1)^n$) (1.4) reduce to the standard forms

$$[t_0, \ldots, t_n]g = \sum_{j=0}^{n} \frac{g(t_j)}{\prod\limits_{l\ne j}(t_j - t_l)}$$

for the finite sequence of knots $\{t_0, \ldots, t_n\} := \{0, \ldots, n\}$ and

$$B_n(x) = \frac{1}{(n-1)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x-k)_+^{n-1} = (-1)^n n[0, 1, \ldots, n](x - \bullet)_+^{n-1}. \tag{1.5}$$

In this article, we will construct a multi-dimensional extension of complex B-splines. This construction uses a geometric interpretation via so-called ridge functions and extends the corresponding known identity in the case of non-negative integral order.

The structure of the paper is as follows. In Section 2, we review some basic properties of Weyl fractional derivatives and integrals in the complex setting. In Section 3, we then use these operators to define complex B-splines for arbitrary knot sequences. Section 4 exhibits some relations between multivariate complex B-splines and Dirichlet averages, also with respect to partial derivatives. In Section 5, we show connections to R-hypergeometric functions by deriving some identities.

2. Weyl fractional derivatives and integrals

In this section, we briefly introduce the Weyl fractional derivative and integral. We will only present those properties that are necessary in what follows. The interested reader is referred to the literature for more details. An incomplete list of references includes [12,19,23,21,24]. We use the notation and terminology given in [19].

We denote by $\mathscr{S}(\mathbb{R})$ the Schwartz space endowed with the usual semi-norms that make it into a Fréchet space.

Definition 2.1. Let $f \in \mathscr{S}(\mathbb{R})$ and let $z \in \mathbb{C}_+$. Then the Weyl fractional integral $W^{-z} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ is defined by

$$(W^{-z} f)(x) := \frac{1}{\Gamma(z)} \int_x^\infty (t-x)^{z-1} f(t)\, \mathrm{d}t.$$

The Weyl fractional derivative $W^z : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ is given by

$$(W^z f)(x) = \frac{(-1)^n}{\Gamma(\nu)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_x^\infty (t-x)^{\nu-1} f(t)\, \mathrm{d}t,$$

with $n = \lceil \operatorname{Re} z \rceil$, and $\nu = n - z$. Here $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$, $x \mapsto \min\{n \in \mathbb{Z} | n \geq x\}$, denotes the *ceiling function*.

On occasion, we also write $f^{(z)}$ for $W^z f$ and $f^{(-z)}$ for W^{-z}.

It is known that both W^z and W^{-z} are linear operators mapping $\mathscr{S}(\mathbb{R})$ into itself [19,24]. Note that some authors define the Weyl fractional derivative and integral only for periodic functions and call W^z and W^{-z} a Riemann–Liouville derivative and integral, respectively. (See, for instance, [24].) In addition, the following rules are obeyed by these two operators. (See, e.g., [10], Chapter XXIII, Section 23.16.)

Proposition 2.2. *Let $f \in \mathscr{S}(\mathbb{R})$ and let $z, \zeta \in \mathbb{C}_+$. Then the operators W^z and W^{-z} satisfy the following identities.*

(1) $W^{-(z+\zeta)} = W^{-z} W^{-\zeta} = W^{-\zeta} W^{-z} = W^{-(\zeta+z)}$;
(2) $W^z W^{-z} = W^{-z} W^z = \mathrm{id}_{\mathscr{S}(\mathbb{R})}$;
(3) $W^{z+\zeta} = W^z W^\zeta = W^\zeta W^z = W^{\zeta+z}$.

Proof. To prove (1), we write

$$\begin{aligned}(W^{-\zeta}(W^{-z}f))(x) &= \frac{1}{\Gamma(\zeta)}\int_x^\infty (t-x)^{\zeta-1}\frac{1}{\Gamma(z)}\int_t^\infty (u-t)^{z-1}\, f(u)\,\mathrm{d}u\,\mathrm{d}t\\ &= \frac{1}{\Gamma(\zeta)}\,\frac{1}{\Gamma(z)}\int_x^\infty\int_t^\infty (t-x)^{\zeta-1}(u-t)^{z-1}\, f(u)\,\mathrm{d}u\,\mathrm{d}t\\ &= \frac{1}{\Gamma(\zeta)}\,\frac{1}{\Gamma(z)}\int_x^\infty\int_x^u (t-x)^{\zeta-1}(u-t)^{z-1}\,\mathrm{d}t\, f(u)\,\mathrm{d}u.\end{aligned}$$

To obtain the last equality, Fubini's Theorem was used. Employing the substitution $t = u - s(u - x)$ yields

$$(W^{-\zeta}(W^{-z}f))(x) = \frac{1}{\Gamma(\zeta)}\,\frac{1}{\Gamma(z)}\int_x^\infty\left(\int_0^1 (1-s)^{\zeta-1}\, s^{z-1}\,\mathrm{d}s\right)(u-x)^{\zeta+z-1}\, f(u)\,\mathrm{d}u.$$

The integral over s produces the two-dimensional beta function $B(\zeta, z)$ and as $B(\zeta, z) = \Gamma(\zeta)\,\Gamma(z)/\Gamma(\zeta+z)$, we are finished. The remaining statements follow from the symmetry of the beta function in its two arguments and the computations above.

For (2), we have, using again Fubini's Theorem and the substitution introduced in (1) above,

$$\begin{aligned}(W^{z}(W^{-z}f))(x) &= \frac{(-1)^n}{\Gamma(n-z)\Gamma(z)}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_x^\infty (t-x)^{n-z-1}\int_t^\infty (u-t)^{z-1}\, f(u)\,\mathrm{d}u\,\mathrm{d}t\\ &= \frac{(-1)^n}{\Gamma(n-z)\Gamma(z)}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_x^\infty\int_x^u (t-x)^{n-z-1}(u-t)^{z-1}\,\mathrm{d}t\, f(u)\,\mathrm{d}u\\ &= \frac{(-1)^n}{\Gamma(n-z)\Gamma(z)}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_x^\infty\left(\int_0^1 (1-s)^{n-z-1}\, s^{z-1}\,\mathrm{d}s\right)(u-x)^{n-1}\, f(u)\,\mathrm{d}u\\ &= \frac{(-1)^n}{\Gamma(n)}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_x^\infty (u-x)^{n-1}\, f(u)\,\mathrm{d}u = f(x).\end{aligned}$$

The last equality follows from differentiation of Cauchy's formula for an n-fold integral:

$$\int_x^\infty \mathrm{d}x_1\int_{x_1}^\infty \mathrm{d}x_2\cdots\int_{x_{n-1}}^\infty f(x_n)\,\mathrm{d}x_n = \frac{(-1)^n}{(n-1)!}\int_x^\infty (t-x)^{n-1} f(t)\,\mathrm{d}t.$$

For the other half of (2), we have

$$\begin{aligned}(W^{-z}(W^{z}f))(x) &= \frac{(-1)^n}{\Gamma(n-z)\Gamma(z)}\int_x^\infty (t-x)^{z-1}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_t^\infty (u-t)^{n-z-1}\, f(u)\,\mathrm{d}u\,\mathrm{d}t\\ &= \frac{(-1)^n}{\Gamma(n-z)\Gamma(z)}\int_x^\infty (t-x)^{z-1}\int_0^\infty u^{n-z-1}\, f^{(n)}(u+t)\,\mathrm{d}u\,\mathrm{d}t.\end{aligned}$$

Now, using first the substitution $v := u + t$, then applying Fubini's Theorem, and finally setting $t = v - s(v - x)$, produces

$$(W^{-z}(W^{z}f))(x) = \frac{(-1)^n}{\Gamma(n)}\int_x^\infty (v-x)^{n-1}\, f^{(n)}(v)\,\mathrm{d}v = f(x),$$

by integration by parts. This shows the validity of (2).

To finally establish (3), note that

$$\mathrm{id}_{\mathscr{S}(\mathbb{R})} = W^{z+\zeta}\, W^{-(z+\zeta)} = W^{z+\zeta}\, W^{-z}\, W^{-\zeta}$$

and

$$\mathrm{id}_{\mathscr{S}(\mathbb{R})} = W^{\zeta}\, W^{z}\, W^{-z}\, W^{-\zeta}.$$

Equating these two identities and observing that W^{-z} maps positive functions to positive functions yields the first half of (3). The second half is obtained by switching the order of ζ and z. □

Note that Proposition 2.2 shows that the system $\{W^{\pm z} | z \in \mathbb{C}_+\}$ forms a multiplicative semi-group. In view of item (2) in Proposition 2.2, we also define $W^0 := \mathrm{id}_{\mathscr{S}(\mathbb{R})}$.

3. Multivariate complex B-splines

In this section, we introduce multivariate complex B-splines using a geometrically inspired definition by means of so-called ridge functions and an identity that is known for splines of non-negative integral orders.

To this end, we recall that for the nth order cardinal B-spline B_n, $n \in \mathbb{N}$, the following relation holds:

$$[0, 1, \ldots, n]g = \frac{1}{n!} \int_{\mathbb{R}} B_n(t) g^{(n)}(t)\, \mathrm{d}t. \tag{3.1}$$

An analog for complex B-splines reads as follows. (See [8].)

Proposition 3.1. *Suppose that* $\operatorname{Re} z > 1$ *and* $g \in \mathscr{S}(\mathbb{R})$. *Then the complex B-spline* B_z *and the complex divided difference* (1.3) *satisfy the following relation.*

$$[z; \mathbb{N}_0]g = \frac{1}{\Gamma(z)} \int_{\mathbb{R}} B_z(t) g^{(z)}(t)\, \mathrm{d}t,$$

where $g^{(z)} := W^z g$ *denotes the Weyl fractional derivative of order* z.

We can extend complex B-splines to include arbitrary weights $b := (b_0, b_1, \ldots) \in \mathbb{C}_+^{\mathbb{N}_0}$ and arbitrary sequences of knots. For this purpose, let Δ^∞ be the infinite-dimensional standard simplex

$$\Delta^\infty := \left\{ u := (u_j)_j \in (\mathbb{R}_0^+)^{\mathbb{N}_0} \,\middle|\, \sum_{j=0}^{\infty} u_j = 1 \right\},$$

endowed with the topology of pointwise convergence, i.e., the weak∗-topology. We denote by $\mu_b = \varprojlim \mu_b^n$ the projective limit of *Dirichlet measures* μ_b^n on the n-dimensional standard simplex Δ^n with density

$$\frac{\Gamma(b_0) \cdots \Gamma(b_n)}{\Gamma(b_0 + \cdots + b_n)}\, u_0^{b_0 - 1}\, u_1^{b_1 - 1} \cdots u_n^{b_n - 1}. \tag{3.2}$$

Definition 3.2. Given a weight vector $b \in \mathbb{C}_+^{\mathbb{N}_0}$ and an increasing knot sequence $\tau := \{\tau_k\}_k \in \mathbb{R}^{\mathbb{N}_0}$ with the property that $\lim_{k\to\infty} \sqrt[k]{\tau_k} \leq \varrho$, for some $\varrho \in [0, e)$, a complex B-spline $B_z(\bullet \mid b; \tau)$ with weight vector b and knot sequence τ is a function satisfying

$$\int_{\mathbb{R}} B_z(t \mid b; \tau) g^{(z)}(t)\, \mathrm{d}t = \int_{\Delta^\infty} g^{(z)}(\tau \cdot u)\, \mathrm{d}\mu_b(u) \tag{3.3}$$

for all $g \in \mathscr{S}^{\omega}(\mathbb{R})$.

Here, $\mathscr{S}^\omega(\mathbb{R}) := \mathscr{S}(\mathbb{R}) \cap C^\omega(\mathbb{R})$, with $C^\omega(\mathbb{R})$ denoting the real-analytic functions on $\mathbb{R}$, and $\tau \cdot u = \sum_{k\in\mathbb{N}} \tau_k u_k$ for $u = \{u_k\}_{k\in\mathbb{N}} \in \Delta^\infty$.

Remark 3.3. For finite $\tau = \tau(n)$ and $b = b(n)$ and $z := n \in \mathbb{N}$, (3.3) defines also the so-called *Dirichlet splines* if g is chosen in $C^n(\mathbb{R})$. For, Dirichlet splines $D_n(\,\cdot \mid b;\tau)$ of order n are defined as those functions for which

$$\int_{\mathbb{R}} g^{(n)}(t) D_n(t|b;\tau)\,\mathrm{d}t = \int_{\Delta^n} g^{(n)}(\tau \cdot u)\,\mathrm{d}\mu_b(u), \quad \tau \in \mathbb{R}^{n+1},$$

holds true for all $g \in C^n(\mathbb{R})$ and thus for $g \in \mathscr{S}^\omega(\mathbb{R})$.

As an analog to (3.1), we define divided differences of g of order z for the sequence of knots τ as

$$[z;\tau]g := \frac{1}{\Gamma(z)} \int_{\mathbb{R}} B_z(t \mid b;\tau) g^{(z)}(t)\,\mathrm{d}t, \quad \text{for all } g \in \mathscr{S}(\mathbb{R}). \tag{3.4}$$

We extend the notion of complex B-splines to a multivariate setting in $\mathbb{R}^s$, $s \geq 1$, via the notion of ridge functions. (See, for instance, [22].) This approach has already led to an extension of the Curry–Schoenberg-splines to a multivariate setting, giving, among others, rise to so-called simplex splines. (See [5,18,20], and also [4].)

To this end, let $\lambda \in \mathbb{R}^s \setminus \{0\}$ be a direction, and let $g : \mathbb{R} \to \mathbb{C}$ be some function. The ridge function corresponding to g is defined as $g_\lambda : \mathbb{R}^s \to \mathbb{C}$,

$$g_\lambda(x) = g(\langle \lambda, x\rangle) \quad \text{for all } x \in \mathbb{R}^s.$$

We denote the canonical inner product in $\mathbb{R}^s$ by $\langle \bullet, \bullet \rangle$ and the norm induced by it as $\| \bullet \|$.

Definition 3.4. Let $\tau = \{\tau^n\}_{n\in\mathbb{N}_0} \in (\mathbb{R}^s)^{\mathbb{N}_0}$ be a sequence of knots in $\mathbb{R}^s$ with the property that

$$\exists\, \varrho \in [0, e) : \limsup_{n\to\infty} \sqrt[n]{\|\tau^n\|} \leq \varrho. \tag{3.5}$$

The multivariate complex B-spline $\boldsymbol{B}_z(\bullet \mid b, \tau) : \mathbb{R}^s \to \mathbb{C}$ with weight vector $b \in \mathbb{C}_+^{\mathbb{N}_0}$ and knot sequence τ is defined by means of the identity

$$\int_{\mathbb{R}^s} g(\langle \lambda, x\rangle) \boldsymbol{B}_z(x \mid b,\tau)\,\mathrm{d}x = \int_{\mathbb{R}} g(t) B_z(t \mid b, \lambda\tau)\,\mathrm{d}t, \tag{3.6}$$

where $g \in \mathscr{S}(\mathbb{R})$, and where $\lambda \in \mathbb{R}^s \setminus \{0\}$ such that $\lambda\tau := \{\langle \lambda, \tau^n\rangle\}_{n\in\mathbb{N}}$ is separated.

Convention 3.5. We will index the elements of a collection of vectors in $\mathbb{R}^s$ or $\mathbb{C}^s$ by superscripts and their components by subscripts. That is, if $T := \{t^1, \ldots, t^n\}$ is a collection of vectors in $\mathbb{C}^s$ then t_j^k denotes the jth component of the kth vector in T.

As the knot set τ depends on z, we write $\tau = \tau(z)$ and note that $\tau(z) = \mathbb{N}_0$ for $z \in \mathbb{C} \setminus \mathbb{N}_0$ and $\tau(z) = \mathbb{N}_n$, for $z \in \mathbb{N}$, where $\mathbb{N}_n := \{0, 1, \ldots, n\}$ denotes the initial segment of $\mathbb{N}_0$ of length $n + 1$. Setting $z := n \in \mathbb{N}$ in (3.6), the infinite sequences b and τ collapse to $b(n) := (b_0, b_1, \ldots, b_n, 0, 0, \ldots)$ and $\tau(n) := (t^1, \ldots, t^n, 0, 0, \ldots)$ and (3.6) becomes a well-known relation between univariate and multivariate B-splines (cf. [11] and [18]).

For the special case $b := e := (1, 1, 1, \ldots)$, the multivariate divided differences of order z are defined on ridge functions via

$$\begin{aligned}[z; \tau]g_\lambda &= [z; \tau]g(\langle\lambda, \bullet\rangle) = \frac{1}{\Gamma(z)} \int_{\mathbb{R}^s} g^{(z)}(\langle\lambda, x\rangle)\boldsymbol{B}_z(x \mid e, \tau)\,\mathrm{d}x \\ &= \frac{1}{\Gamma(z)} \int_{\mathbb{R}} g^{(z)}(t) B_z(t \mid e, \lambda\tau)\,\mathrm{d}t = [z; \lambda\tau]g, \quad \forall g \in \mathscr{S}(\mathbb{R}^\infty),\end{aligned}$$

where $\lambda \in \mathbb{R}^s \setminus \{0\}$ such that $\lambda\tau$ is separated.

Ridge functions form a dense subset of $C(\mathbb{R}^k)$, $k \in \mathbb{N}$. For a fixed direction this was shown in [17] and the general case is due to [16]. Let $n \in \mathbb{N}$, let $\tau := \{\tau^0, \tau^1, \ldots, \tau^n\}$ be a finite sequence of knots, and let $e_n := (\underbrace{1, \ldots, 1}_{n+1\text{-times}}, 0, \ldots)$. Then,

$$\begin{aligned}[\tau^0, \ldots, \tau^n]g_\lambda &:= [n; \tau]g(\langle\lambda, \bullet\rangle) = \frac{1}{\Gamma(z)} \int_{\mathbb{R}^s} g^{(n)}(\langle\lambda, x\rangle)\boldsymbol{B}_n(x \mid e_n, \tau)\,\mathrm{d}x \\ &= \frac{1}{\Gamma(z)} \int_{\mathbb{R}} g^{(n)}(t) B_n(t \mid e_n, \lambda\tau)\,\mathrm{d}t = [n; \lambda\tau]g = \sum_{j=0}^{n} \frac{g(\langle\lambda, \tau^j\rangle)}{\prod_{l\neq j} \langle\lambda, \tau^j - \tau^l\rangle}.\end{aligned}$$

In order to derive some identities of multivariate complex B-splines, we need to introduce *Dirichlet averages* and discuss some of their properties.

4. Dirichlet averages

Dirichlet averages are discussed in the book by Carlson [1] and related to univariate and multivariate B-splines in [2]. Dirichlet averages have produced deep and interesting connections to special functions. In this section, we extend the notion of the Dirichlet average to the infinite-dimensional setting and show that under mild conditions on the weights, the results important for our interests do also hold on Δ^∞. In particular, we show that using a geometric interpretation, the Weyl fractional derivative and integral can be applied to Dirichlet averages.

To this end, let Ω be an open convex subset of $\mathbb{C}^s$, $s \in \mathbb{N}$, let $\tau \in \times_{i=0}^{n} \Omega$, and let $b \in \mathbb{C}_+^{n+1}$. Then the Dirichlet average of a measurable function $f : \Omega \to \mathbb{C}$ is defined as the integral

$$F(b; \tau) := \int_{\Delta^n} f(\tau \cdot u)\mathrm{d}\mu_b^n(u), \tag{4.1}$$

where $\tau \cdot u := \sum_{i=0}^{n} u_i \tau^i \in \mathbb{C}^s$ and μ_b^n denotes the Dirichlet measure with density given by (3.2). We note that it is customary to denote the Dirichlet average of a function f by the corresponding upper-case letter, F. It can be shown that the Dirichlet average of a derivative equals the derivative of the Dirichlet average. (For more details regarding the properties of Dirichlet averages and their connection to the theory of special functions, we refer the interested reader to the work by Carlson [1].)

The following result is known.

Proposition 4.1. *Suppose that $f : \Omega \to \mathbb{C}$ is holomorph. Then the Dirichlet average $F(\cdot, \tau)$ is a holomorphic function on $\mathbb{C}_+^{n+1}$, for fixed $\tau \in \Omega^{n+1}$.*

Proof. See [1], Theorem 5.2.-2. □

The extension of (4.1) to Δ^∞ consists of taking Ω to be an open convex set in $\mathbb{C}^s$, $b \in \mathbb{C}_+^{\mathbb{N}_0}$, and choosing a measurable function $f \in \mathscr{S}(\Omega) := \mathscr{S}(\Omega, \mathbb{C})$. For $\tau \in \Omega^{\mathbb{N}_0} \subset (\mathbb{C}^s)^{\mathbb{N}_0}$ and $u \in \Delta^\infty$, define $\tau \cdot u$ to be the bilinear mapping $(\tau, u) \mapsto \sum_{i=1}^\infty u_i \tau^i$. The infinite sum exists whenever

$$\limsup_{n\to\infty} \sqrt[n]{\|\tau^n\|} \leq \varrho, \quad \text{some } \varrho \in [0, e), \tag{4.2}$$

where $\|\cdot\|$ now denotes the canonical Euclidean norm on $\mathbb{C}^s$. (See also [8].)

Definition 4.2. Let $f : \Omega \subset \mathbb{C}^s \to \mathbb{C}$ be a measurable function. The Dirichlet average $F : \mathbb{C}_+^{\mathbb{N}_0} \times \Omega^{\mathbb{N}_0} \to \mathbb{C}$ over Δ^∞ is defined by

$$F(b; \tau) := \int_{\Delta^\infty} f(\tau \cdot u)\, \mathrm{d}\mu_b(u),$$

where $\mu_b = \varprojlim \mu_b^n$ is the projective limit of Dirichlet measures on the n-dimensional standard simplex Δ^n.

Remark 4.3. The existence of μ_b is also discussed in [13,26,14,15]. For a detailed discussion of the measure μ_b and the simplex Δ^∞, we also refer to [9] and [8].

Under the assumption that $f : \Omega \to \mathbb{C}$ is a holomorphic function, $b \in \mathbb{C}_+^{\mathbb{N}_0}$ and $\tau \in \Omega^{\mathbb{N}_0}$ satisfies (4.2), the Dirichlet average on Δ^∞ exists and is holomorph on $\mathbb{C}_+^\infty$ for fixed τ. Using the fact that Δ^∞ is the projective limit of its finite-dimensional projections Δ^n, $n \in \mathbb{N}$, the following known properties of F extend naturally to the infinite-dimensional setting. These are summarized in the proposition below.

Proposition 4.4. *The Dirichlet average over Δ^∞ enjoys the following properties.*

(1) *Invariance under permutations: If $\sigma : \mathbb{N}_0^\infty \to \mathbb{N}_0^\infty$ is a permutation, then*

$$F(b_{\sigma(0)}, b_{\sigma(1)}, \ldots, ; \tau^{\sigma(0)}, \tau^{\sigma(1)}, \ldots) = F(b_0, b_1, \ldots; \tau^0, \tau^1, \ldots);$$

(2) *Multiple knots collapse and increase the weights:*

$$F(b_0, b_1, b_2, \ldots; \tau^1, \tau^1, \tau^2, \ldots) = F(b_0 + b_1, b_2, \ldots; \tau^1, \tau^2, \ldots);$$

(3) *Zero weights can be omitted:*

$$F(0, b_1, b_2, \ldots; \tau^0, \tau^1, \tau^2, \ldots) = F(b_1, b_2, \ldots; \tau^1, \tau^2, \ldots);$$

(4) *Single knot of infinite multiplicity reproduces f: If $\tau = (\tau_0, \tau_0, \tau_0, \ldots) \in \Omega^{\mathbb{N}_0}$, then $\tau \cdot u = \tau_0 \sum_{i=0}^\infty u_i = \tau_0 \in \mathbb{C}^s$, and thus $F(b; \tau) = f(\tau_0)$.*

Now suppose that the weight vector $b \in \ell^1(\mathbb{N}_0)$. Let $c := \sum_{i=0}^\infty b_i$ and $w_i := \frac{b_i}{c}$. Since for $j = 1, \ldots, n$, the equation

$$\int_{\Delta^n} u_j \mathrm{d}\mu_b^n(u) = \frac{b_j}{\sum\limits_{i=0}^n b_i}$$

holds for all finite-dimensional projections Δ^n of Δ^∞ [1], we have

$$\int_{\Delta^\infty} u_j \mathrm{d}\mu_b(u) = \frac{b_j}{c} = w_j, \quad \forall j \in \mathbb{N}_0.$$

Using the definition of Dirichlet measure (3.2) and the fact that, for finite $m := (m_0, m_1, \ldots, m_k) \in \mathbb{N}^{k+1}$ and $b := (b_0, b_1, \ldots, b_l) \in \mathbb{C}_+^{k+1}$ (Equation (8) in Section 4.4 in [1]),

$$u_0^{m_0} u_1^{m_1} \cdots u_k^{m_k} \mathrm{d}\mu_b^k(u) = \frac{B(b+m)}{B(b)} \mathrm{d}\mu_{b+m}^k(u), \tag{4.3}$$

where $B : \mathbb{C}^{k+1} \setminus H^{k+1} \to \mathbb{C}$, with

$$H^{k+1} := \left\{ z := (z_0, z_1, \ldots, z_k) \in \mathbb{C}^{k+1} \,\middle|\, \sum_{i=0}^{k} z_i \in \mathbb{Z}_0^- \right\},$$

denotes the multi-dimensional beta function

$$B(z) := B(z_0, z_1, \ldots, z_k) := \frac{\prod_{i=0}^{k} \Gamma(z_i)}{\Gamma\left(\sum_{i=0}^{k} z_i\right)},$$

one obtains the identity

$$u_j \, \mathrm{d}\mu_b(u) = w_j \, \mathrm{d}\mu_{b+e_j}(u), \quad j \in \mathbb{N}_0, \tag{4.4}$$

generalizing the corresponding finite-dimensional identity [1]. (Here $e_j := \{\delta_{i,j} | i \in \mathbb{N}_0\}$.) Using that $\sum_{j \in \mathbb{N}_0} u_j = 1$, one obtains from (4.4) the identity

$$\int_{\Delta^\infty} f(\tau \cdot u) \mathrm{d}\mu_b(u) = \sum_{j=0}^{\infty} w_j \int_{\Delta^\infty} f(\tau \cdot u) \mathrm{d}\mu_{b+e_j}(u),$$

or, equivalently,

$$F(b; \tau) = \sum_{j=0}^{\infty} w_j F(b + e_j; \tau).$$

In particular, for $x \in \mathbb{R}^s$ and $g(x) := x f(x)$, this last equation gives

$$G(b; \tau) = \sum_{j=0}^{\infty} w_j \tau^j F(b + e_j; \tau),$$

where $\tau^j \in \mathbb{C}^s$ is the jth component of τ. (See also [2], Equation 5.6.)

The results regarding the relations between Dirichlet averages found in [2], Section 5, or [20], Section 3, transfer to the infinite-dimensional setting using the definition of projective limit. We omit further details.

Of particular interest are Weyl fractional derivatives of Dirichlet averages and their relation to the Dirichlet averages of Weyl fractional derivatives. To this end, let Ω be again an open convex subset of $\mathbb{C}^s$ and $f \in \mathscr{S}(\Omega)$. Let $z := (z_1, \ldots, z_s)^\top \in \mathbb{C}_+^s$ and let $n_i := \lceil \operatorname{Re} z_i \rceil$ and $\nu_i := n_i - z_i$, $i = 1, \ldots, s$. Furthermore, let $x := (x_1, \ldots, x_s)^\top \in \Omega$. The Weyl partial fractional derivative $\partial_{x_i}^z$ with respect to x_i, $i = 1, \ldots, s$, of order z is defined by

$$\partial_{x_i}^z f(x) := \frac{\partial^z}{\partial x_i^z} f(x) := \frac{(-1)^n}{\Gamma(\nu)} \frac{\partial^n}{\partial x_i^n} \int_{(\mathbb{R}_0^+)^s} (t - x)^{\nu - 1} f(t) \mathrm{d}t,$$

where $(t - x)^{\nu-1} = (t_1 - x_1)^{\nu-1} \cdot \ldots \cdot (t_s - x_s)^{\nu-1}$ and $\Gamma(\nu) = \Gamma(\nu_1) \cdot \ldots \cdot \Gamma(\nu_s)$.

Consider for a moment the case $s = 1$. Let $\tau \in \times_{i=0}^{k} \Omega$ and denote by $\partial_i := \frac{\partial}{\partial \tau_i}$, $i = 0, 1, \ldots, k$, the partial derivative operator. Let $f : \Omega \to \mathbb{C}$ be of class C^1 and consider the operator $\sum_{i=0}^{k} \partial_i = \langle \nabla, e(k) \rangle$, where ∇ is the k-dimensional gradient and $e(k) = (1, \ldots, 1) \in \mathbb{N}^k$. However, $\langle \nabla, e(k) \rangle f$ is equal to the *univariate* derivative $\frac{\mathrm{d}}{\mathrm{d}x} f(x, \ldots, x)$, where $x := \tau_1 = \cdots = \tau_k$. This suggests the following definition.

Definition 4.5. Let $g \in \mathscr{S}(\Omega)$ and let $z \in \mathbb{C}_+$. Then

$$\left(\sum_{i=0}^{k} \partial_i \right)^z g(\tau) := \frac{(-1)^n}{\Gamma(\nu)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_{\mathbb{R}_+} t^{\nu-1} g(x+t, \ldots, x+t) \mathrm{d}t$$
$$= \frac{(-1)^n}{\Gamma(\nu)} \int_{\mathbb{R}_+} t^{\nu-1} g^{(n)}(x+t, \ldots, x+t)\, \mathrm{d}t,$$

where $\tau_1 = \cdots = \tau_k = x$.

Now, set $g(\tau) := F(b(k); \tau)$, with $b(k) \in \mathbb{C}_+^{k+1}$ a finite weight vector. Then, by the properties of Dirichlet averages and the fact that $\sum_{i=1}^{k} u_i = 1$, one obtains with $\partial(k)^z := \left(\sum_{i=0}^{k} \partial_i \right)^z$,

$$(\partial(k)^z F(b(k); \bullet))(\tau) = \frac{(-1)^n}{\Gamma(\nu)} \int_{\mathbb{R}_0^+} t^{\nu-1} F^{(n)}(b(k); x+t, \ldots, x+t) \mathrm{d}t$$
$$= \frac{(-1)^n}{\Gamma(\nu)} \int_{\mathbb{R}_0^+} t^{\nu-1} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_{\Delta^k} f(u_1(x+t) + \cdots + u_k(x+t)) \mathrm{d}\mu_{b(k)}^k(u)\, \mathrm{d}t$$
$$= \frac{(-1)^n}{\Gamma(\nu)} \int_{\mathbb{R}_0^+} t^{\nu-1} \int_{\Delta^k} f^{(n)}(u_1(x+t) + \cdots + u_k(x+t))(u_1 + \cdots + u_k)^n \mathrm{d}\mu_{b(k)}^k(u)\, \mathrm{d}t$$
$$= \int_{\Delta^k} (\partial(k)^z f)(u \cdot \tau) \mathrm{d}\mu_{b(k)}^k(u).$$

In a quite similar fashion, one shows that

$$(\partial_i^z F(b(k); \bullet))(\tau) = \int_{\Delta^k} u_i^z f^{(z)}(u \cdot \tau) \mathrm{d}\mu_{b(k)}^k(u),$$

and, if $\{i_1, \ldots, i_m\} \subseteq \{0, 1, \ldots, k\}$, $m = 0, 1, \ldots, k$,

$$(\partial_{i_1}^{z_{i_1}} \cdots \partial_{i_m}^{z_{i_m}} F(b(k); \bullet))(\tau) = \int_{\Delta^k} u_{i_1}^{z_{i_1}} \cdots u_{i_m}^{z_{i_m}} f^{(z_{i_1} + \cdots + z_{i_m})}(u \cdot \tau) \mathrm{d}\mu_{b(k)}^k(u),$$

where $z_{i_\ell} \in \mathbb{C}_+$, $\ell = 1, \ldots, m$.

In order to extend the above results to Δ^∞, we need to consider the operator $\partial := \partial(\infty) := \sum_{i=0}^{\infty} \partial_i$, where ∂_i denotes again the partial derivative with respect to τ_i, $i \in \mathbb{N}_0$.

Remark 4.6. Operators of the type considered above naturally act on functions $f : \mathbb{R}^\infty \to \mathbb{C}$ for which, for instance, the semi-norm $|f|_{1,\infty} := \sum_{i \in \mathbb{N}_0} \|\partial_i f\|_\infty$ is finite. In other words, the function f can be regarded as an element of the Sobolev space $W^{1,\infty}(\mathbb{R}^\infty)$, defined as the projective limit of the Sobolev spaces $W^{1,\infty}(\mathbb{R}^n)$: $W^{1,\infty}(\mathbb{R}^\infty) := \varprojlim W^{1,\infty}(\mathbb{R}^n)$. In the current setting, however, these ideas will not be pursued further.

Instead, we consider the following scenario. Let $f \in \mathscr{S}(\mathbb{R}^\infty)$ and let $z \in \mathbb{C}_+$ with $n := \lceil \operatorname{Re} z \rceil$ and $\nu := n - z$. Define $\partial^z := \left(\sum_{i=0}^\infty \partial_i\right)^z$ to be the operator on $\mathscr{S}(\mathbb{R}^\infty)$ given by the expression

$$(\partial^z f)(\tau) := \frac{(-1)^n}{\Gamma(\nu)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_0^\infty t^{\nu-1} f(x+t)\mathrm{d}t = \frac{(-1)^n}{\Gamma(\nu)} \int_0^\infty t^{\nu-1} f^{(n)}(x+t)\mathrm{d}t,$$

where $\mathbb{R} \ni x := \tau_1 = \tau_2 = \cdots = \tau_n = \cdots$. Replacing f by the Dirichlet average $G(b; \bullet)$ of some function $g \in \mathscr{S}(\mathbb{R}^\infty)$ and for a weight vector $b \in \mathbb{C}_+^{\mathbb{N}_0} \cap \ell^1(\mathbb{N}_0)$, one obtains by arguments similar to those given above that

$$(\partial^z G(b; \bullet))(\tau) = \int_{\Delta^\infty} g^{(z)}(u \cdot \tau)\mathrm{d}\mu_b(u),$$

and

$$(\partial_i^{z_i} G(b; \bullet))(\tau) = \int_{\Delta^\infty} u_i^{z_i} g^{(z_i)}(u \cdot \tau)\mathrm{d}\mu_b(u),$$

and also

$$(\partial_{i_1}^{z_{i_1}} \cdots \partial_{i_m}^{z_{i_m}} G(b; \bullet))(\tau) = \int_{\Delta^\infty} u_{i_1}^{z_{i_1}} \cdots u_{i_m}^{z_{i_m}} g^{(z_{i_1}+\cdots+z_{i_m})}(u \cdot \tau)\mathrm{d}\mu_b(u), \tag{4.5}$$

for any $\{i_1, \ldots, i_m\} \subseteq \mathbb{N}_0$.

Our next goal is the generalization of some results, in particular (3.18), (3.19), and Theorem 2, presented in [28] regarding the fractional Weyl integral and derivative of Dirichlet averages to complex orders.

To this end, let $b \in \mathbb{C}_+^{\mathbb{N}_0} \cap \ell^1(\mathbb{N}_0)$ and $z \in \mathbb{C}_+^{\mathbb{N}_0} \cap \ell^1(\mathbb{N}_0)$. To proceed, we need the following lemma.

Lemma 4.7. *For $k \in \mathbb{N}_0$ and $b, z \in \mathbb{C}_+^{\mathbb{N}_0} \cap \ell^1(\mathbb{N}_0)$ define finite segments of b and z by $b(k) := (b_0, b_1, \ldots, b_k, 0, \ldots)$ and $z(k) := (z_0, z_1, \ldots, z_k, 0, \ldots,)$, respectively. Then the ratio of beta functions*

$$\frac{B(b(k) + z(k))}{B(b(k))} \tag{4.6}$$

remains finite when $k \to \infty$.

Proof. Rewriting (4.6) in terms of the gamma function, we obtain

$$\frac{B(b(k) + z(k))}{B(b(k))} = \frac{\Gamma\left(\sum_{i=0}^k b_i\right) \prod_{i=0}^k \Gamma(b_i + z_i)}{\prod_{i=0}^k \Gamma(b_i)\, \Gamma\left(\sum_{i=0}^k b_i + z_i\right)}.$$

Employing the following identity (cf. (2), p. 5, of [6])

$$\frac{\Gamma(u)}{\Gamma(u+v)} = \mathrm{e}^{\gamma\, v} \prod_{n=0}^\infty \left(1 + \frac{v}{u+n}\right) \mathrm{e}^{-v/(n+1)},$$

where γ denotes the Euler–Mascheroni constant, the above expression can be rewritten as

$$\frac{B(b(k)+z(k))}{B(b(k))} = \prod_{n=0}^{\infty} \frac{1+\frac{z_0+\cdots+z_k}{b_0+\cdots+b_k+n}}{\left(1+\frac{z_0}{b_0+n}\right)\cdot\ldots\cdot\left(1+\frac{z_k}{b_k+n}\right)}$$
$$= \prod_{n=0}^{\infty} \frac{n+\sum\limits_{i=0}^{k}(b_i+z_i)}{n+\sum\limits_{i=0}^{k} b_i} \prod_{i=0}^{k}\left(1-\frac{z_i}{n+b_i+z_i}\right).$$

Note that since b and z are assumed to be in $\ell^1(\mathbb{N}_0)$, the sum $\sum_{i=0}^{\infty}(b_i+z_i)$ converges. Moreover, for a fixed $n \in \mathbb{N}_0$, the infinite product $\prod_{i=0}^{\infty}\left(1-\frac{z_i}{n+b_i+z_i}\right)$ converges absolutely iff the sum $\sum_{i=0}^{\infty}\left|-\frac{z_i}{n+b_i+z_i}\right|$ converges. This, however, is guaranteed if the sequence

$$\Sigma_1(b,z) := \left\{\frac{z_i}{z_i+b_i}\,\middle|\, i\in\mathbb{N}_0\right\}, \tag{4.7}$$

is in $\ell^1(\mathbb{N}_0)$, for then

$$\sum_{i=0}^{\infty}\frac{z_i}{b_i+z_i+n} \le \sum_{i=0}^{\infty}\frac{z_i}{b_i+z_i} < \infty, \quad \forall n\in\mathbb{N}_0.$$

As for any fixed $k \in \mathbb{N}_0$ the infinite product over $n \in \mathbb{N}_0$ converges absolutely, we deduce that, under the condition that b, z and $\{z_i/(z_i+b_i) | i \in \mathbb{N}_0\}$ are in $\ell^1(\mathbb{N}_0)$, the double infinite product

$$\prod_{n=0}^{\infty}\prod_{i=0}^{\infty} \frac{n+\sum\limits_{i=0}^{\infty}(b_i+z_i)}{n+\sum\limits_{i=0}^{\infty} b_i}\left(1-\frac{z_i}{n+b_i+z_i}\right) \tag{4.8}$$

converges absolutely. □

In the following, we denote the double product (4.8) by $\beta_1(b,z)$. Clearly,

$$\beta_1(b(k),z(k)) = \frac{B(b(k)+z(k))}{B(b(k))}, \quad k\in\mathbb{N}_0.$$

Denoting the projective limit of the measures $\{u_0^{z_0}\,u_1^{z_1}\cdots u_k^{z_k}\,\mathrm{d}\mu_b^k(u)\}$ as $k\to\infty$ by $\widetilde{\mu}_b$, we have that

$$\widetilde{\mu}_b = \beta_1(b,z)\,\mu_{b+z}.$$

Here and in the following, we define the addition $b+z$ of two quantities such as b and z componentwise.

Thus, since (4.4) is valid for all finite $k \in \mathbb{N}_0$, we obtain, using the properties of projective limit, for the Weyl derivative of order $z \in C_+^{\mathbb{N}_0}$ of the Dirichlet average G (with respect to the measure $\widetilde{\mu}_b$) of a function $g \in \mathscr{S}(\mathbb{R}^\infty)$

$$(W^z G(b;\bullet))(\tau) = \beta_1(b,z)\,G^{(z)}(b+z;\tau), \tag{4.9}$$

where we set $W^z := \prod_{i=0}^{\infty} \partial_i^{z_i}$. The left hand side of (4.9) is to be interpreted as the projective limit uniquely determined by the finite-dimensional projections $\prod_{i=0}^{k} \partial_i^{z_i}$, $k \in \mathbb{N}_0$.

Eq. (4.9) generalizes the result in the corollary to Theorem 1 in [28] to complex orders and the infinite-dimensional setting. We summarize these findings in the theorem below.

Theorem 4.8. *Suppose that $b \in \mathbb{C}_+^{\mathbb{N}_0} \cap \ell^1(\mathbb{N}_0)$ and $z \in \mathbb{C}_+^{\mathbb{N}_0} \cap \ell^1(\mathbb{N}_0)$. Further suppose that the sequence $\Sigma_1(b, z)$ given by (4.7) is in $\ell^1(\mathbb{N}_0)$ and that $g \in \mathscr{S}(\mathbb{R}^\infty)$. Then, the Weyl derivative W^z of order $z \in \mathbb{C}_+^{\mathbb{N}_0}$ of the Dirichlet average $G(b; \bullet)$ of g is given by*

$$(W^z G(b; \bullet))(\tau) = \beta_1(b, z)\, G^{(z)}(b + z; \tau).$$

To obtain a similar result for Weyl fractional integrals of Dirichlet averages, we note that the proof of Theorem 1 in [28] completely transfers to the (finite-dimensional) complex setting with $z \in \mathbb{C}$ satisfying $0 < \operatorname{Re} z < 1$. However, quantities such as $(i\,\cdot)^z$ need now be interpreted as $|\cdot|^z\, e^{i\pi z \operatorname{sgn}(\cdot)/2}$. Here, we used the signum function $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$ defined by

$$\operatorname{sgn}(x) := \begin{cases} -1, & x < 0; \\ 0, & x = 0; \\ +1, & x > 0. \end{cases}$$

For the sake of completeness and reference, we restate this theorem adapted to our setting.

Theorem 4.9. *Let $k \in \mathbb{N}$. Suppose that $f \in \mathscr{S}(\mathbb{R})$ and that $z \in \mathbb{C}_+^{k+1}$ satisfies $0 < \operatorname{Re} z_i < 1$, for all $i = 0, 1, \ldots, k$. Further, assume that $b \in \mathbb{R}_+^{k+1}$ is such that $\operatorname{Re} z < b$. Then*

$$(W^{-z} F(b; \bullet))(\tau) = \frac{B(b - \operatorname{Re} z)}{B(b)}\, (W^{-z} F)(b - \operatorname{Re} z; \tau), \tag{4.10}$$

where, for $z := (z_0, \ldots, z_k) \in \mathbb{C}_0^{k+1}$,

$$(W^{-z} f)(t) := (W^{-(z_0,\ldots,z_k)} f)(t) := \frac{1}{\Gamma(z)} \int_{(\mathbb{R}_0^+)^{k+1}} (v - t)^{z-1} f(v) \mathrm{d}v,$$

with $(v - t)^{z-1} = (v_0 - t_0)^{z_0 - 1} \cdot \ldots \cdot (v_k - t_k)^{z_k - 1}$.

In order to extend the result in Theorem 4.9 to the infinite-dimensional setting and to complex weights, we need to investigate the convergence properties of the ratio of beta functions $B(b - z)/B(b)$, $b - z \notin H^{k+1}$. For this purpose, we need the next lemma.

Lemma 4.10. *Suppose that $b, z \in \mathbb{C}_+^{\mathbb{N}_0} \cap \ell^1(\mathbb{N}_0)$ and that $b - z \notin H^{k+1}$. For $k \in \mathbb{N}_0$, let $z(k) := (z_0, \ldots, z_k)$ and $b(k) := (b_0, \ldots, b_k)$. Then the ratio of beta functions*

$$\frac{B(b(k) - z(k))}{B(b(k))} \tag{4.11}$$

remains finite as $k \to \infty$.

Proof. We rewrite (4.11) as

$$\frac{B(b(k)-z(k))}{B(b(k))} = \frac{B(b(k)-z(k))}{B(b(k)-z(k)+z(k))} = \frac{\prod_{i=0}^{k}\Gamma(b_i-z_i)}{\Gamma\left(\sum_{i=0}^{k}(b_i-z_i)\right)}\,\frac{\Gamma\left(\sum_{i=0}^{k}b_i\right)}{\prod_{i=0}^{k}\Gamma(b_i)},$$

and apply again identity (2) on p. 5 of [6]. After some algebra, this yields

$$\frac{B(b(k)-z(k))}{B(b(k))} = \prod_{n=0}^{\infty}\frac{n+\sum_{i=0}^{k}(b_i-z_i)}{n+\sum_{i=0}^{k}b_i}\prod_{i=0}^{k}\left(1+\frac{z_i}{b_i-z_i+n}\right). \tag{4.12}$$

Under the assumption that $b, z \in \mathbb{C}_+^{\mathbb{N}_0} \cap \ell^1(\mathbb{N}_0)$, the right hand side of (4.12) converges absolutely if in addition $\sum_{i=0}^{\infty}\frac{z_i}{b_i-z_i}$ converges absolutely. Henceforth, we thus also assume that the sequence

$$\Sigma_2(b,z) := \left\{\frac{z_i}{b_i-z_i}\,\middle|\, i\in\mathbb{N}_0\right\} \tag{4.13}$$

is in $\ell^1(\mathbb{N}_0)$. Thus, both infinite products converge absolutely, implying the absolute convergence of the double infinite product

$$\beta_2(b,z) := \prod_{n=0}^{\infty}\prod_{i=0}^{\infty}\frac{n+\sum_{i=0}^{\infty}(b_i-z_i)}{n+\sum_{i=0}^{\infty}b_i}\left(1+\frac{z_i}{b_i-z_i+n}\right). \quad \square$$

As before, we of course have for finite $b(k) := (b_0, b_1, \ldots, b_k, 0, \ldots,)$ and finite $z(k) := (z_0, z_1, \ldots, z_k, 0, \ldots)$

$$\beta_2(b(k), z(k)) = \frac{B(b(k)-z(k))}{B(b(k))}, \quad k\in\mathbb{N}_0.$$

By virtue of the projective limit definition, we therefore obtain the sought-after generalization of Theorem 1 in [28] to the infinite-dimensional setting and complex orders $z \in \mathbb{C}_+^{\mathbb{N}_0}$ with $0 < \operatorname{Re} z < 1$ and $b \in \mathbb{C}_+^{\mathbb{N}_0}$ with $b - z \notin H^\infty := \left\{z := (z_0, z_1, \ldots, z_i, \ldots) \in \mathbb{C}^{\mathbb{N}_0} \mid \sum_{i=0}^{\infty} z_i \in \mathbb{Z}_0^-\right\}$. Thus, we have:

Theorem 4.11. *Suppose that $f \in \mathscr{S}(\mathbb{R})$ and $z \in \mathbb{C}_+^{\mathbb{N}_0} \cap \ell^1(\mathbb{N}_0)$ with $0 < \operatorname{Re} z < 1$. Furthermore, suppose that $b \in \mathbb{C}_+^{\mathbb{N}_0} \cap \ell^1(\mathbb{N}_0)$ is such that $b - z \notin H^\infty$, and that the sequence $\Sigma_2(b,z)$ defined in (4.13) is an element of $\ell^1(\mathbb{N}_0)$. Then*

$$(W^{-z}F(b;\bullet))(\tau) = \beta_2(b,z)\,(W^{-z}F)(b-z;\tau).$$

Now, assume that $a, b \in \mathbb{C}_+^{\mathbb{N}_0} \cap \ell^1(\mathbb{N}_0)$ and $z \in \mathbb{C}_+$. Then,

$$\begin{aligned}
\int_{\mathbb{R}} f^{(z)}(u)\, B_z(u|b;\tau)\,\mathrm{d}u &= (\partial^z F(b;\bullet))(\tau) = (W^{b-a}W^{-(b-a)}\partial^z F(b;\bullet))(\tau)\\
&= W^{-(b-a)}\left(\beta_2(b,b-a)(W^{b-a}\partial^z F)(b-a;\bullet)\right)(\tau)\\
&= \beta_2(b,b-a)\, W^{-(b-a)}\left(\int_{\Delta^\infty}(W^{b-a}\, f^{(z)})(u\cdot\bullet)\mathrm{d}\mu_{b-a}(u)\right)(\tau)\\
&= \beta_2(b,b-a)\, W^{-(b-a)}\left(\int_{\mathbb{R}}(W^{b-a}\, f^{(z)})(u)\, B_z(u\mid b-a;\bullet)\,\mathrm{d}u\right)(\tau)\\
&= \beta_2(b,b-a)\int_{\mathbb{R}}(W^{b-a}\, f^{(z)})(u)\,(W^{-(b-a)}B_z)(u\mid b-a;\tau)\,\mathrm{d}u.
\end{aligned}$$

Given a $\lambda \in \mathbb{R}^s \setminus \{0\}$, choose $\{t^n | n \in \mathbb{N}_0\}$ such that $\tau_i = \langle\lambda, t^i\rangle$, for all $i \in \mathbb{N}_0$. Then, with $\tau = \{\langle\lambda, t^n\rangle | n \in \mathbb{N}_0\}$, the last equation above reads as follows:

$$\begin{aligned}
&\int_{\mathbb{R}}(W^{b-a}\, f^{(z)})(u)\,(W^{-(b-a)}B_z)(u\mid b-a;\tau)\,\mathrm{d}u\\
&\quad= \int_{\mathbb{R}}(W^{b-a}\, f^{(z)})(u)\,(W^{-(b-a)}B_z)(u\mid b-a;\lambda t)\,\mathrm{d}u\\
&\quad= \int_{\mathbb{R}^s}(W^{b-a}\, f^{(z)})(\langle\lambda,x\rangle)\,(W^{-(b-a)}\boldsymbol{B}_z)(x\mid b-a;t)\,\mathrm{d}x.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_{\mathbb{R}} f^{(z)}(u)\, B_z(u\mid b;\tau)\,\mathrm{d}u\\
&\quad= \beta_2(b,b-a)\int_{\mathbb{R}}(W^{b-a}\, f^{(z)})(u)\,(W^{-(b-a)}B_z)(u\mid b-a;\lambda t)\,\mathrm{d}u\\
&\quad= \beta_2(b,b-a)\int_{\mathbb{R}^s}(W^{b-a}\, f^{(z)})(\langle\lambda,x\rangle)\,(W^{-(b-a)}\boldsymbol{B}_z)(x\mid b-a;t)\,\mathrm{d}x. \qquad (4.14)
\end{aligned}$$

Identity (4.14) provides a generalization of the result given in Theorem 2 in [28], in particular, the extension to multivariate complex B-splines.

5. Some identities for multivariate complex B-splines

Returning to the general case $s > 1$, the results obtained in the previous section apply to $\zeta \in \Omega^{\mathbb{N}_0} \subset (\mathbb{C}^s)^{\mathbb{N}_0}$ by considering partial derivatives with respect to the components ζ_i^j of $\zeta^j \in \zeta$. (See also [2] for the finite-dimensional vectorial setting.)

We assume that the weight vector $b \in \mathbb{C}_+^{\mathbb{N}_0} \cap \ell^1(\mathbb{N}_0)$. Let $\lambda \in \mathbb{R}^s \setminus \{0\}$ be a direction and let $z \in \mathbb{C}_+$. Furthermore, assume that $\tau := \{\tau^n\}_{n\in\mathbb{N}_0}$ is a knot sequence in $\mathbb{R}^s$ satisfying condition (4.2) with $\lambda\tau = \{\langle\lambda, \tau^n\rangle\}_{n\in\mathbb{N}}$. Employing Theorem 3 in [2] or Theorem 3.1 in [20] for the functions $g^{(z)} \in \mathscr{D}(\mathbb{R}) \subset \mathscr{S}(\mathbb{R})$ and $g_j^{(1+z)} := (\langle\lambda, \tau^j\rangle - \bullet)g^{(1+z)}$, $j \in \mathbb{N}_0$, yields for their Dirichlet averages on the knot sequence $\lambda\tau$

$$(c-1)G^{(z)}(b;\lambda\tau) = (c-1)G^{(z)}(b-e_j;\lambda\tau) + G_j^{(1+z)}(b;\lambda\tau), \qquad (5.1)$$

where G_j is the Dirichlet average of g_j, and $c = \sum_{i\in\mathbb{N}_0} b_i$ as above, for a weight vector $b \in \ell^1(\mathbb{N}_0)$. Applying (3.3) to (5.1) we obtain

$$(c-1)\int_{\mathbb{R}} g^{(z)}(t)B_z(t \mid b; \lambda\tau)\mathrm{d}t = (c-1)\int_{\mathbb{R}} g^{(z)}(t)B_z(t \mid b-e_j; \lambda\tau)\mathrm{d}t + \int_{\mathbb{R}} (\langle\lambda, \tau^j\rangle - t)g^{(1+z)}(t)B_z(t \mid b; \lambda\tau)\mathrm{d}t.$$

This last equation, however, is by the defining equation of multivariate complex B-splines $\boldsymbol{B}_z$ equivalent to

$$(c-1)\int_{\mathbb{R}^s} g^{(z)}(\langle\lambda, x\rangle)\boldsymbol{B}_z(x \mid b; \tau)\mathrm{d}x = (c-1)\int_{\mathbb{R}^s} g^{(z)}(\langle\lambda, x\rangle)\boldsymbol{B}_z(x \mid b-e_j; \tau)\mathrm{d}x + \int_{\mathbb{R}^s} \langle\lambda, \tau^j - x\rangle\, g^{(1+z)}(\langle\lambda, x\rangle)\boldsymbol{B}_z(x \mid b; \tau)\mathrm{d}x, \quad j \in \mathbb{N}_0. \tag{5.2}$$

We summarize these results in a theorem:

Theorem 5.1. *Let $\tau := \{\tau^n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^s$ be a knot sequence satisfying condition (3.5), and let $b \in \ell^1(\mathbb{N}_0)$ be a weight vector. Assume that $\lambda \in \mathbb{R}^s \setminus \{0\}$ and $z \in \mathbb{C}_+$. Furthermore, assume that $g^{(z)} \in \mathscr{D}(\mathbb{R}^\infty)$ and let $g_j^{(1+z)} := (\tau^j - \bullet)g^{(1+z)}$, $j \in \mathbb{N}_0$. Then*

$$(c-1)\int_{\mathbb{R}^s} g_\lambda^{(z)}(x)\boldsymbol{B}_z(x \mid b; \tau)\mathrm{d}x = (c-1)\int_{\mathbb{R}^s} g_\lambda^{(z)}(x)\boldsymbol{B}_z(x \mid b-e_j; \tau)\mathrm{d}x + \int_{\mathbb{R}^s} \langle\lambda, \tau^j - x\rangle\, g_\lambda^{(1+z)}(x)\boldsymbol{B}_z(x \mid b; \tau)\mathrm{d}x, \quad j \in \mathbb{N}_0.$$

Now suppose that $\tau = \{\tau^k\}_{k\in\mathbb{N}_0}$ is such that its convex hull $\operatorname{conv}\tau$ does not contain $0 \in \mathbb{R}^s$, and let $n \in \mathbb{N}$. Following [1], we define the R-hypergeometric function $R_a(b;\tau) : \mathbb{R}_+^{n+1} \times \Omega^{n+1} \to \mathbb{C}$ by

$$R_a(b;\tau) := \int_{\Delta^n} (\tau \cdot u)^a \mathrm{d}\mu_b^n(u),$$

where $\Omega := H$, H a half-plane in $\mathbb{C} \setminus \{0\}$, if $a \in \mathbb{C} \setminus \mathbb{N}$, and $\Omega := \mathbb{C}$, if $a \in \mathbb{N}$. It can be shown (see [1]) that R_{-a}, $a \in \mathbb{C}_+$, has a holomorphic continuation in τ to $\mathbb{C}_0$, where $\mathbb{C}_0 := \{\zeta \in \mathbb{C} \mid -\pi < \arg\zeta < \pi\}$.

Since this result holds for all $n \in \mathbb{N}$, the definition and properties of R_{-a} can be lifted to the infinite-dimensional simplex Δ^∞ using the properties of the projective limit, provided that the above-given conditions on τ and the weight vector b are satisfied. Using (3.3), we can express R_{-a} as follows. Firstly, we require a result from the Weyl fractional differentiation theory (see Section 2.2 in [12]), which states that for $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha \geq 0$, and $\beta \in \mathbb{C}$

$$(W^{-\alpha}t^{\beta-1})(x) = \frac{\Gamma(1+\alpha-\beta)}{\Gamma(1-\beta)}\, x^{\beta-\alpha-1}, \tag{5.3}$$

provided that $\operatorname{Re}(\alpha+\beta-\lfloor\operatorname{Re}\alpha\rfloor) < 1$. Here $\lfloor\cdot\rfloor : \mathbb{R} \to \mathbb{Z}$, $x \mapsto \max\{n \in \mathbb{Z} \mid n \leq x\}$, denotes the *floor function*.

Now suppose that $z \in \mathbb{C}$ is such that $\operatorname{Re} z > 1$ and choose an $a \in \mathbb{C}_+$. Then, by virtue of (5.3), we can write

$$t^{-a} = \frac{\Gamma(a)}{\Gamma(a-z)}\,[t^{-(a-z)}]^{(z)},$$

provided $\operatorname{Re} a > 2\operatorname{Re} z - \lfloor \operatorname{Re} z \rfloor > 1$. Hence, with (3.3),

$$R_{-a}(b;\tau) = \int_{\Delta^\infty} (\tau\cdot u)^{-a}\mathrm{d}\mu_b(u) = \frac{\Gamma(a)}{\Gamma(a-z)}\int_{\mathbb{R}} (t^{-(a-z)})^{(z)} B_z(t \mid b;\tau)\mathrm{d}t, \tag{5.4}$$

for an $a \in \mathbb{C}$ satisfying $\operatorname{Re} a > 2\operatorname{Re} z - \lfloor \operatorname{Re} z \rfloor$.

Assume that $\lambda \in \mathbb{R}^s \setminus \{0\}$ is a direction and that the knot sequence $\tau = \{t^k\}_{k\in\mathbb{N}_0}$ satisfies $\langle \lambda, t^k\rangle < 1$, for all $k \in \mathbb{N}_0$. Then,

$$\begin{aligned} R_{-a}(b;1-\lambda\tau) &= \frac{\Gamma(a)}{\Gamma(a-z)}\int_{\mathbb{R}} (t^{-(a-z)})^{(z)} B_z(t \mid b;1-\lambda\tau)\mathrm{d}t \\ &= \frac{\Gamma(a)}{\Gamma(a-z)}\int_{\mathbb{R}} [(1-t)^{-(a-z)}]^{(z)} B_z(t \mid b;\lambda\tau)\mathrm{d}t \\ &= \frac{\Gamma(a)}{\Gamma(a-z)}\int_{\mathbb{R}^s} [(1-\langle\lambda,x\rangle)^{-(a-z)}]^{(z)} \boldsymbol{B}_z(x \mid b;\tau)\mathrm{d}x. \end{aligned}$$

Here, $1-\lambda\tau$ is defined componentwise: $1-\lambda\tau = \{1-\langle\lambda,\tau^n\rangle \mid n\in\mathbb{N}_0\}$. Hence, we proved the following theorem, which is a generalization of the classical Euler-type integral representation for the Gauss hypergeometric function.

Theorem 5.2. *Suppose that $z \in \mathbb{C}$ with $\operatorname{Re} z > 1$ and $a \in \mathbb{C}$ are such that $\operatorname{Re} a > 2\operatorname{Re} z - \lfloor \operatorname{Re} z \rfloor$. Moreover, let $\lambda \in \mathbb{R}^s \setminus \{0\}$ be such that $\langle\lambda,\tau^k\rangle < 1$, for all $k \in \mathbb{N}_0$. Then the R-hypergeometric function R_{-a} can be expressed as*

$$R_{-a}(b;1-\lambda\tau) = \frac{\Gamma(a)}{\Gamma(a-z)}\int_{\mathbb{R}^s} [K_{a-z}(\langle\lambda,x\rangle)]^{(z)} \boldsymbol{B}_z(x \mid b;\tau)\mathrm{d}x, \tag{5.5}$$

where $K_{a-z} := (1-\bullet)^{-(a-z)}$.

Let us recall the following formula for the R-hypergeometric function R_{-a} which is an extension of the finite-dimensional setting (see Theorem 6.8-3 in [1]) to the infinite-dimensional case under the assumption that the knots $\zeta := \{\zeta_n\}_{n\in\mathbb{N}_0}$, $\zeta_n > 0$ for all $n \in \mathbb{N}_0$, satisfy (4.2), and that the weight vector b is an element of $\ell^1(\mathbb{N}_0)$.

$$R_{-a}(b;\zeta) = \prod_{n=0}^{\infty} \zeta_n^{-b_n}\, R_{a-c}(b;\zeta^{-1}), \tag{5.6}$$

where $c = \sum_{i=0}^{\infty} b_i$, with $c \notin -\mathbb{N}_0$, and $\zeta^{-1} := \{\zeta_n^{-1}\}_{n\in\mathbb{N}_0}$.

Now, choosing weights b_n, $n \in \mathbb{N}_0$, so that setting $a := c \in \mathbb{R}$ still satisfies the condition $a > 2\operatorname{Re} z - \lfloor \operatorname{Re} z \rfloor$, and using the fact that $R_0 = 1$, we obtain from (5.5) and (5.6),

$$\int_{\mathbb{R}^s} [K_{a-z}(\langle\lambda,x\rangle)]^{(z)} \boldsymbol{B}_z(x \mid b;\tau)\mathrm{d}x = \frac{\Gamma(a-z)}{\Gamma(a)}\prod_{n=0}^{\infty}(1-\langle\lambda,\tau^n\rangle)^{-b_n},$$

which is a generalization of Watson's Identity. (Cf. [27] and [20].)

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